ON THE USE OF THE ENTROPY PRINCIPLE IN GENERAL RELATIVITY

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Abstract

The main purpose of this article is to use the extension of thermodynamics to general relativity, previously proposed by the author, to obtain expressions which will give the criteria for the thermodynamic equilibrium of a static gravitating system in a readily applicable mathematical form. After restating the principle chosen by the author as the general relativity analogue of the second law of thermodynamics, and showing once more that it is a natural covariant generalization of the ordinary second law of thermodynamics, the principle is then applied to finite systems in general and to adiabatic systems having no flux of matter or heat at the boundary. The mathematical conditions for thermodynamic equilibrium are then obtained for the case of any finite static system, and a specially useful form for these conditions is obtained for the case that the system has spherical symmetry.

§1. Introduction

In several previous articles, I have attempted an extension of thermodynamics to general relativity[1] and have tried to make certain applications of this extension.[2] The main purpose of the present article is to prepare for certain further applications which I propose to make.

The postulate, which must be taken in work of this kind as the general relativity analogue of the ordinary first law of thermodynamics, is evidently Einstein's generalized energy-momentum principle, which reduces in flat space-time to the ordinary laws of the conservation of energy and momentum, and is a principle which certainly must not be violated by any thermodynamic changes which may take place. The use of this principle has already been discussed in some detail in the preceding article in this journal,[3] and, those conclusions drawn from it, which will be necessary for the present considerations.

The postulate which was chosen as the general relativity analogue of the second law of thermodynamics was a new one, which was justified, so far as may be, by showing it to be a natural covariant generalization of the ordinary second law of thermodynamics in flat space-time. In the present article we shall be especially interested in the method of applying this principle to determine the equilibrium conditions for a finite system in a static state.

[1] Tolman, Proc. Nat. Acad. **14**, 268 (1928); ibid. **14**, 701 (1928).
[2] Tolman, Proc. Nat. Acad. **14**, 348 (1928); ibid. **14**, 353 (1928).
[3] Tolman, Phys. Rev. **35**, 875 (1930).
[4] See reference (3) last paragraph of §10.

In the immediately following section, §2, we shall again state the postulate chosen as the general relativity analogue of the second law of thermodynamics in the form which it assumes for an infinitesimal region, and show again by way of review that it is a natural covariant generalization of the ordinary second law of thermodynamics valid in flat space-time. In §3, we shall obtain by integration the form taken by the principle in the case of a finite system in general, and then more especially in the case of an adiabatic system having no flux of matter or heat at the boundary. Following this we shall consider, in §4, the conditions for thermodynamic equilibrium in the case of a finite static system, and finally, in §5, we shall consider a specific form in which these conditions for equilibrium can be put in case the system under consideration has spherical symmetry.

§2. The Generalized Second Law of Thermodynamics

To state the postulate which was taken as the general relativity analogue of the second law of thermodynamics, we shall first define the *entropy vector* at a given point in space-time, by the equation

$$S^{\mu} = \phi_0 \frac{dx_{\mu}}{ds} \tag{1}$$

where ϕ_0 is the *proper density of entropy* at the point in question as measured by a local observer, and dx_{μ}/ds is the *macroscopic velocity* of matter at that point. Corresponding to this vector, we also have the *entropy vector density* given by the equation

$$\mathfrak{S}^{\mu} = S^{\mu}\sqrt{-g} = \phi_0\sqrt{-g}\,\frac{dx_{\mu}}{ds}\,. \tag{2}$$

The second law postulate can then be stated in the form

$$\frac{\partial \mathfrak{S}^{\mu}}{\partial x_{\mu}} dx_1 dx_2 dx_3 dx_4 \geqq \frac{dQ_0}{T_0} \tag{3}$$

where dQ_0 is the *heat measured in proper coordinates* flowing through the boundary into the infinitesimal region and during the infinitesimal time denoted by $dx_1 dx_2 dx_3 dx_4$, and T_0 is the *proper temperature* at the boundary.

To show that expression (3) is covariant, we rewrite it by a well-known transformation in the form

$$(S^{\mu})_{\mu}\sqrt{-g}dx_1 dx_2 dx_3 dx_4 \geqq \frac{dQ_0}{T_0}\,. \tag{4}$$

The quantity $(S^{\mu})_{\mu}$, however, is the contracted covariant derivative of S^{μ} and is known to be an invariant, while $\sqrt{-g}dx_1 dx_2 dx_3 dx_4$ and dQ_0/T_0 are also obviously invariants so that both sides of expression (4) are tensors of rank zero, and the requirement of covariance is met in the simplest possible way.

To show that expression (3) reduces to the ordinary requirements of the second law in the limiting case of flat space-time, we first rewrite it with the help of (2) in the form

$$\frac{\partial}{\partial x_\mu}\left(\phi_0\sqrt{-g}\,\frac{dx_\mu}{ds}\right)dx_1dx_2dx_3dx_4 \geqq \frac{dQ_0}{T_0}\cdot \tag{5}$$

In flat space-time, however, using Galilean coordinates x, y, z and t, we shall have $\sqrt{-g}=1$, and writing out the indicated summation in full, we obtain with some rearrangement in the order of terms

$$\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)dxdydzdt$$

$$\geqq -\left[\frac{\partial}{\partial x}\left(\phi_0\frac{dx}{ds}\right)+\frac{\partial}{\partial y}\left(\phi^0\frac{dy}{ds}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dz}{ds}\right)\right]dxdydzdt + \frac{dQ_0}{T_0}$$

and this can evidently be rewritten as

$$\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)dxdydzdt$$

$$\geqq\left[\frac{\partial}{\partial x}\left(\phi_0\frac{dt}{ds}\,\frac{dx}{dt}\right)+\frac{\partial}{\partial y}\left(\phi_0\frac{dt}{ds}\,\frac{dy}{dt}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dt}{ds}\,\frac{dz}{dt}\right)\right]dxdydzdt + \frac{dQ_0}{T_0}. \tag{6}$$

In accordance, however, with the special theory of relativity, valid in flat space-time, entropy is an invariant for the Lorentz transformation, so that entropy density will be affected by the Lorentz-Fitzgerald factor of contraction ds/dt in such a way that we can make the substitution

$$\phi=\phi_0\frac{dt}{ds} \tag{7}$$

where ϕ is the density of entropy in the particular set of Galilean coordinates that are being used. Furthermore, since heat and temperature have the same transformation factors on the basis of the special theory, we may also substitute

$$\frac{dQ}{T}=\frac{dQ_0}{T_0}\cdot \tag{8}$$

Hence, using u, v, w to denote the components of ordinary velocity dx/dt etc. we can finally rewrite expression (6) in the form

$$\frac{\partial\phi}{\partial t}dxdydzdt \geqq -\left[\frac{\partial}{\partial x}(\phi u)+\frac{\partial}{\partial y}(\phi v)+\frac{\partial}{\partial z}(\phi w)\right]dxdydzdt + \frac{dQ}{T}\cdot \tag{9}$$

This expression, however, evidently states the essence of the ordinary second law of thermodynamics, since it requires that the actual increase of entropy occurring in the time dt in the infinitesimal volume $dx\ dy\ dz$ cannot be less than the total entropy brought in from the outside by the flux of matter and the flow of heat.

Our postulate as to the general relativity analogue of the second law has thus been justified both by the fact of its expression in covariant form and by its reduction in the limiting case of flat space-time to the ordinary second law of thermodynamics. The justification, thus presented, is of course in no sense a complete proof of the validity of the postulate taken. Covariance and agreement with the form of the second law valid in the special theory of relativity are necessary but not sufficient requirements for determining the new form of the second law. The new postulate must be regarded as a real generalization containing an element not present in the special theory of relativity, and the ultimate complete justification of the postulate must be dependent on the agreement between the conclusions that can be drawn from it and actual experimental or observational facts.

§3. Application of the Entropy Principle to Finite Systems

To apply the new principle to the changes taking place in a finite system, let us start with the principle in the form given by expression (5), and taking x_1, x_2 and x_3 as being the space-like coordinates, integrate over the spatial region of interest. If we carry out such an integration, using coordinates such that the limits of integration necessary to include the whole system fall on the actual boundary which separates the system from its surroundings, it is evident that the summation of dQ_0/T_0 over the interior of the system will cancel out, since any heat entering a given element of volume is abstracted from neighboring elements, so that we shall only have to consider the heat entering the system from its surroundings. Hence dividing equation (5) through by dx_4, writing out the indicated summation, and performing the suggested integration, we obtain with some rearrangement in the order of terms

$$\begin{aligned}\frac{\partial}{\partial x_4}\iiint\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \\ \geqq -\iiint\left[\frac{\partial}{\partial x_1}\left(\phi_0\sqrt{-g}\,\frac{dx_1}{ds}\right)+\frac{\partial}{\partial x_2}\left(\phi_0\sqrt{-g}\,\frac{dx_2}{ds}\right)\right. \\ \left.+\frac{\partial}{\partial x_3}\left(\phi_0\sqrt{-g}\,\frac{dx_3}{ds}\right)\right]dx_1dx_2dx_3+\sum\left(\frac{1}{T_0}\,\frac{dQ_0}{dx_4}\right)_{\text{Boundary}}\end{aligned}$$

The last term on the right hand side of this expression is the total value taken over the boundary of the system of the quantity $(1/T_0)$ (dQ_0/dx_4), and by performing the indicated integrations the other terms on this side of the expression can also be seen to depend solely on quantities whose values are determined by conditions at the boundary, provided we continue to use, as suggested above, a set of coordinates so chosen that the limits of integration necessary to include the whole of the system actually lie on the boundary. We obtain

$$\frac{\partial}{\partial x_4}\iiint\left(\phi_0\sqrt{-g}\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \geqq \iint\left|\phi_0\sqrt{-g}\frac{dx_1}{ds}\right|_{x_1}^{x_1'}dx_2dx_3$$
$$-\iint\left|\phi_0\sqrt{-g}\frac{dx_2}{ds}\right|_{x_2}^{x_2'}dx_1dx_3-\iint\left|\phi_0\sqrt{-g}\frac{dx_3}{ds}\right|_{x_3}^{x_3'}dx_1dx_2 \tag{10}$$
$$+\sum\left(\frac{1}{T_0}\frac{dQ_0}{dx_4}\right)_{\text{Boundary}}$$

where the limits of integration at the boundary of the system are denoted by x_1, x_1' etc.

This expression (10) may be regarded as a general statement of the second law of thermodynamics as applied to finite systems, and defining the entropy of the system by the equation

$$S=\iiint\left(\phi_0\sqrt{-g}\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \tag{11}$$

the expression can be interpreted as giving the relation which must hold between the rate at which the entropy of a finite system is changing with the "time" x_4, and those conditions existing at the boundary which determine the flux of matter and the flow of heat.

For the case of an adiabatic system with no flow of heat through the boundary, and in addition with the quantities dx_1/ds, dx_2/ds and dx_3/ds equal to zero at the boundary, so that there is no flux of matter between the system and its surroundings, expression (10) reduces to

$$\frac{\partial}{\partial x_4}\iiint\left(\phi_0\sqrt{-g}\frac{dx_4}{ds}\right)dx_1dx_2dx_3 \geqq 0. \tag{12}$$

In accordance with this expression, the entropy of an adiabatic system of the kind described can only increase or remain constant with increases in the time x_4.

§4. Thermodynamic Equilibrium in a Static System

We shall now investigate the conditions for thermodynamic equilibrium in a static system. To do this we must examine the changes which could take place from one static state to another without violating either the energy-momentum principle or the entropy principle as applied to the system as a whole.

Consider a system which together with its surroundings is originally in some given static state, such that none of the components $g_{\mu\nu}$ of the metrical tensor are changing with the time, and furthermore such that there is no flow of heat nor macroscopic flux of matter or radiation at any point. Without alteration in the metric or the distribution of matter and radiation outside of the system we then assume some change to take place in the distribution of matter and radiation inside the boundary in such a way that the system ultimately arrives in some new possible static state. Assuming no detailed knowledge of the exact nature of the internal process which occurs, we now inquire into the restrictions which the energy-momentum principle and the entropy principle applied to the system as a whole would impose on the change in state.

In accordance with a conclusion obtained in my previous article on the energy-momentum principle,[4] the restrictions imposed by this principle on the possible changes in line element within the system are to be met by the condition that the components $g_{\mu\nu}$ of the metrical tensor and their first differential coefficients $\partial g_{\mu\nu}/\partial x_\alpha$ are to retain their values unaltered at the boundary. These restrictions, coming from considerations of the energy-momentum principle, must be applied as part of the thermodynamic criteria for determining the possible changes in state.

Turning now to the entropy principle, we note that the process under consideration has been so chosen that the change is an adiabatic one of the kind described in the preceding section, since by hypothesis the flow of heat and macroscopic flux of matter are everywhere zero at the start and remain so at the boundary while the process is under way; hence we can at once apply expression (12) to the process. We thus obtain as the restriction imposed by the entropy principle the requirement that the entropy as defined by equation (11) shall not be decreased by the process. This restriction must be applied as giving the remaining thermodynamic criteria for determining the possible changes in state.

The considerations of the last two paragraphs immediately make it evident that the condition for the thermodynamic equilibrium of a static system, with no flow of heat or flux of matter at any point, is that the entropy of the system as given by equation (11) shall be the maximum that can be obtained without violating the boundary conditions furnished from considerations of the energy-momentum principle. Stating this conclusion more specifically by the use of the calculus of variations, we may now give as the condition of thermodynamic equilibrium in a static system of the kind described above

$$\delta \int\int\int \phi_0 \sqrt{-g}\frac{dx_4}{ds} dx_1 dx_2 dx_3 = 0 \tag{13}$$

under the subsidiary condition holding *at the boundary of the system*

$$\delta g_{\mu\nu} = \delta\left(\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) = 0 \qquad 14)$$

where it is to be remembered that the conclusions have been derived using a system of coordinates such that the limits of integration necessary to include the whole of the system of interest fall on the actual boundary which separates the system from its surroundings.

§5. Equilibrium in a Static System Having Spherical Symmetry

As just mentioned the conditions given by equations (13) and (14) have been derived using a system of coordinates such that the limits of integration necessary to include the whole of the system fall on the actual boundary separating it from its surroundings. In the case of a system having spherical symmetry, it is possible, however, to translate these results into a system of polar coordinates, and thus put them into a more convenient form for use.

Let us take a static system of the kind discussed above, with no flow of heat nor macroscopic flux of matter or radiation at any point, and having spherical spatial symmetry, and let us initially make use of a set of coordinates x, y, z, t with the center of symmetry at the origin of the three equivalent spatial axes x, y and z. Such a set of coordinates is of the kind used in deriving the restrictions given by equations (13) and (14).

The line element for our system in these coordinates can evidently be written in the form

$$ds^2 = -e^{\mu}\,(dx^2+dy^2+dz^2) + e^{\nu}dt^2 \tag{15}$$

where the exponents μ and ν are independent of the time t, and on account of the spherical symmetry depend on the coordinates x, y and z in such a way that they are expressible as functions of $(x^2+y^2+z^2)^{1/2}$.

In accordance with this line element we have

$$\begin{gathered} g_{11}=g_{22}=g_{33}=-e^{\mu} \qquad g_{44}=e^{\nu} \\ \sqrt{-g}=e^{\frac{3\mu+\nu}{2}} \end{gathered} \tag{16}$$

and since there is no macroscopic flux of matter or radiation, we have the value zero for all the macroscopic velocities dx_{μ}/ds except for the case $\mu=4$, and then have

$$\frac{dx_4}{ds} = \frac{dt}{ds} = e^{-\nu/2}. \tag{17}$$

Substituting the values given by equations (16) and (17) into equations (13) and (14) we then have as the requirement for thermodynamic equilibrium in our present coordinates

$$\delta\int\!\!\int\!\!\int \phi_0 e^{3\mu/2}dxdydz=0 \tag{18}$$

under the subsidiary condition holding at the boundary of the system

$$\delta\mu = \delta\left(\frac{\partial\mu}{\partial x_{\alpha}}\right) = \delta\nu = \delta\left(\frac{\partial\nu}{\partial x_{\alpha}}\right) = 0. \tag{19}$$

These requirements, however, can now easily be translated into polar coordinates r, θ and ϕ by setting

$$\begin{aligned} x&=r\sin\theta\cos\phi \\ y&=r\sin\theta\sin\phi \\ z&=r\cos\theta \end{aligned} \tag{20}$$

and noting that the condition of spherical symmetry makes it possible to take μ and ν as functions of r alone

$$\mu=\mu(r) \qquad \nu=\nu(r). \tag{21}$$

The requirements for thermodynamic equilibrium then evidently become

$$\delta \iiint \phi_0 e^{3\mu/2} r^2 \sin\theta\, dr d\vartheta d\phi = 0 \tag{22}$$

under the subsidiary condition

$$\delta\mu=\delta\mu'=\delta\nu=\delta\nu'=0 \tag{23}$$

where the primes indicate differentiation with respect to r, and the limitation given by (23) is to be applied at the actual boundary separating the system from its surroundings rather than at the limits of integration which must be given to the new variables in order to include the region of interest.

Finally if we take the region of interest as being a spherical shell contained between the constant radii r_1 and r_2, we can evidently rewrite equations (22) and (23) in the form

$$\delta\left[4\pi \int_{r_1}^{r_2} \phi_0 e^{3\mu/2}\, r^2 dr\right]=0 \tag{24}$$

under the subsidiary condition

$$\delta\mu=\delta\mu'=\delta\nu=\delta\nu'=0 \quad (\text{at } r_1 \text{ and } r_2). \tag{25}$$

It is believed that this form of the conditions of equilibrium will be found an easy and useful one to employ.